Random Matrix Ensembles: Numerical Experiments & Observations based on Random Matrices

A Project Report

Submitted in partial fulfillment of the requirements for the award of the degree of

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In

Industrial Mathematics & Informatics

Submitted by

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Candidate's Declaration

I hereby declare that the work, which is being presented in this project report entitled **Random Matrix Ensembles: Numerical Experiments & Observations based on Random Matrices**, in fulfillment of Summer Project while pursuing **Master of Science in Industrial Mathematics & Informatics** submitted in the **Department of Mathematics, Indian Institute of Technology Roorkee**. This record is an authentic record of my own work carried out from May, 2013 under supervision of **Dr. V. Murugesan, Supercomputer Education and Research Center, Indian Institute of Science, Bangalore, India**. The matter embodied in this report has not been submitted by me for the award of any other degree in this Institute or any other Institute/University.

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Certificate

This is to certify that the dissertation report entitled **Random Matrix Ensembles: Numerical Experiments & Observations based on Random Matrices**, prepared by **Mr. Akash Saha**, in partial fulfillment for the award of degree of Master of Science in Industrial Mathematics & Informatics from Indian Institute of Technology Roorkee is a record of his own work carried out under my supervision and guidance.

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Abstract

This report is based on the observations and the conclusions that could be drawn about random matrix ensembles and their eigenvalues. The relation between complex random matrices and circular law has also been studied. Propositions regarding the behaviour of eigenvalues of random matrices where entries follow normal or uniform distributions have also been stated and computationally put forward using MATLAB. The circular law for complex symmetric (non-hermitian) random matrix has also been studied thoroughly. Some results based on tridiagonal & Hessenberg reduction of real random matrix ensembles with entries normally distributed have been studied and extended for a general case with mean zero and arbitrary non-zero variance.

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Chapter 1

Wigner's Semi-circle Law

Eugene Wigner had studied the distribution of the eigenvalue of a particular kind of matrix known as Wigner matrix. A Wigner matrix is defined as a random matrix $A_{i,j}$ is a $n \times n$ matrix where n is natural number and the entries

- $A_{i,j}, i < j$ are iid (real or complex valued)
- $A_{i,i}$ are iid real random variables (which could possibly follow a different distribution)
- $A_{i,j} = \overline{A}_{j,i}\ \forall 1 \leq i, j \leq n$
- $E[A_{i,i+1}] = 0$, $E[\mid A_{i,i+1} \mid^2] = 1$, $E[A_{j,j}] = 0$, $E[{A^2}_{j,j}] = 1$, where $1 \leq i < n$ and $1 \leq j \leq n$.

1.1 Empirical Spectral Distribution

An Empirical Spectral Distribution of a square matrix A with real eigenvalues is the probability measure of the corresponding eigenvalues λ_i, for $i = 1, \cdots, n$

$$\mu_A(x) = \sum_{k=1}^{n} \chi(w : \lambda_k < x)\,, \text{where } \chi \text{ is the indicator function}$$

For a square matrix A with complex eigenvalues the Empirical Spectral Distribution (ESD) is the probability measure of the corresponding eigenvalues λ_i, for $i = 1, \cdots, n$

$$\mu_A(x,y) = \frac{1}{n}|\{1 \leq i \leq n, Re\,\lambda_i \leq x, Im\,\lambda_i \leq y\}|$$

The ESDs of random matrices are of great importance as their behavior when $n \to \infty$ as then it converges to specific distributions for different classes of random matrices.The following law illustrates the fact about the significance of ESDs

1.2 Semi-circle Law

The semi-circle law was one of the first crucial results related to random matrices.

Theorem 1. *Let X_n be the $n \times n$ Wigner random matrix. Then the ESD of $\frac{1}{\sqrt{n}}X_n$ converges to the semi-circle distribution.*

The semi-circle distribution can be defined as follows

Semi-circle Distribution:- A semi-circle distribution is the probability distribution with density

$$\frac{1}{2\pi}\sqrt{4-x^2} \text{ on } [-2,2]$$

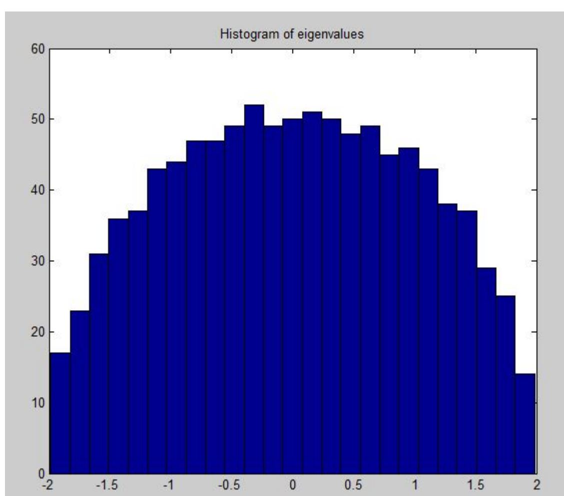

Figure 1.1: Histogram of eigenvalues of real random symmetric matrices of size 1000 with entries following Normal (0,1) Distribution as an illustration of the Semicircle Law

The proof for this theorem was first given by Wigner in [15]. There are proofs also given in [1]. The proofs available essentially involves using any of these two methods

1. The method of moments

2. The method of Stieltjes' transforms

A method using the invariance principle could also be used to which is a general probabilistic technique that deduces the limit of the eigenvalues of Wigner matrices.

1.3 Gaussian Ensembles

Gaussian ensembles refers to the random matrices whose entries follow a Gaussian (or Normal) distribution. The Gaussian ensembles that are of great importance are the Gaussian Orthogonal Ensembles (GOE) and the Gaussian Unitary Ensembles (GUE) which are defined as

Gaussian Ensembles: Let A be a square matrix with entries iid $N(0,1)$ and B be a square matrix with entries iid $CN(0,1)$ then matrix $X = \dfrac{A+A^T}{2}$ is called a GOE and $Y = \dfrac{B+B^T}{2}$ is called a GUE. Both GOEs and GUEs have real eigenvalues as they are all hermitian in nature. As the GOEs and GUEs can also be constructed as Wigner matrices therfore they satisfy the semi-circle law. The joint distribution of the eigenvalues of a GOE matrix could be computed precisely which was given by Mehta in [10].

Theorem 2. *The joint probability density function for the eigenvalues of matrices from a Gaussian orthogonal or Gaussian unitary ensemble of size N is given by*

$$P_N(x_1, x_2, \cdots, x_N) = C_N\, exp\left(-\frac{1}{2}\beta \sum_{j=1}^{N} x_j^2\right) \prod_{j<k} |x_j - x_k|^2$$

where $\beta = 1$ for GOE and $\beta = 2$ for GUE. The constant C_N is determined in such a way that P_N is normalized to unity

According to Selberg the normalization constant C_N is given by

$$C_N^{-1} = (2\pi)^{N/2}\beta^{-N/2-\beta N(N-1)/4}[\Gamma(1+\beta/2)]^{-N}\prod_{j=1}^{N}\Gamma(1+\beta j/2)$$

Chapter 2

Circular Law

The spectral distribution for non-hermitian random matrices with complex entries converges to a uniform distribution on a circular disc was proposed by Girko [6] as

Proposition 1 (Circular Law Conjecture). *Let X_n be the $n \times n$ random matrix whose entries are iid complex (non-hermitian) random variables with mean 0 and variance 1. Then the ESD of $\frac{1}{\sqrt{n}}X_n$ converges (both in probability and in the almost sure sense) to the uniform distribution on the unit disc around origin.*

Although a lot of proofs have been given by many but most of them use some additional assumptions but still the above Circular Law remains a conjecture

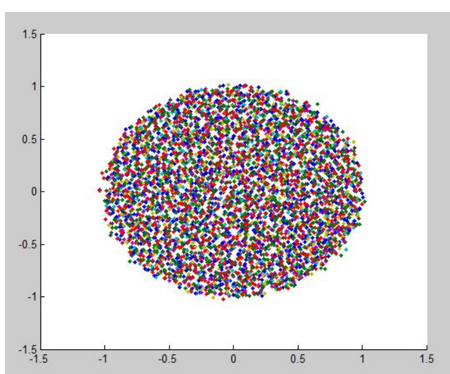

Figure 2.1: Plot of eigenvalues of 10 random matrices whose entries have real & imaginary part as iid N(0,1) and 500 size after being scaled by $\sqrt{2\,size}$ illustrating the Circular Law Conjecture. The factor $\sqrt{2}$ is required as entries are made iid $CN(0,1)$ by taking real and imaginary part both as iid $N(0,1)$.

The Circular Law Conjecture has been found to be true for complex random matrices with entries following normal or uniform distributions with non-zero mean and non-unity variance. It was also observed based on numerical evidences that for entries following non-zero mean resulted in two discs being formed one around the origin and the other formed by largest eigenvalue of each matrices whose center depends on the mean of the distribution of the entries.

2.1 Problems Faced in Proving Circular Law

The circular law for complex matrices could be imagined as the normal extension of the Wigner semi-circle law as it the semi-circle gets extended to form a full circle as the entries of the matrices have both real and imaginary components. But the proof becomes non-trivial because of the following reasons

- Truncation leads to errors: A small change in the values of a random matrix might produce a large change in their eigenvalues thereby rendering truncation methods to be redundant.

- Failure of the Moment Method: The method of moments used successfully for Hermitian matrices but it fails for general complex matrices as the claims about the distribution of a complex matrix do not have any simple form to determine their distribution by uniquely determining their mixed moments of some matrix form of original matrix ensemble.

- Difficulty in Method of Stieltjes' transforms: Some results utilizing Stieltjes' transforms used in case of Wigner matrices do not hold through for proving the circular law. Problems regarding the bounds and mathematical analysis of convergence of Stieltjes' transforms seem to be the shortcomings of this method in proving the circular law.

2.2 Interesting Observations Based on Circular Law

Although the circular law has not been proved like the Wigner semi-circle law but it provides interesting observations and some useful conclusions to be drawn from it. Behaviour of complex random matrix ensembles by varying the nature of the entries of the matrices have been mentioned below.

When the entries of the random matrix ensemble are complex numbers with real and imaginary part iid Normal distribution $N(\mu, \sigma^2)$ (say) then it could be observed that the largest eigenvalue (in magnitude) for each random matrix considered is distributed in a smaller circular disc around the value

$$n\,\mu\,(1+i)$$

in the argand plane, where n is the size of the random matrix. The case is same when

(a) (b)

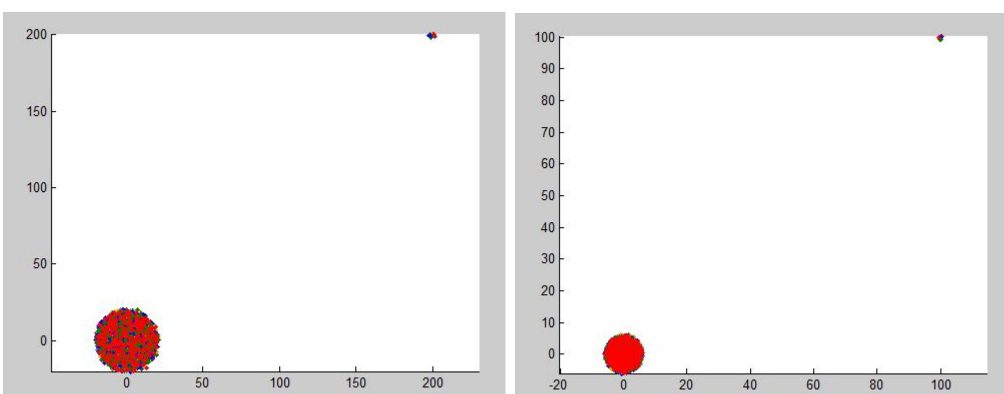

Figure 2.2: (a) Plot of eigenvalues of 10 random matrices whose entries have real & imaginary part as iid $N(1,1)$ and 200 size showing the central disc and the disc of largest eigenvalues centered around (200,200). (b) Plot of eigenvalues of 10 random matrices whose entries have real & imaginary part as iid $U(0,1)$ and 200 size showing the central disc and the disc of largest eigenvalues centered around (100,100).

the real and imaginary part of the random matrix entries are iid Uniform distribution in (a,b) where the mean is $\mu = \frac{a+b}{2}$. The nature of the spectral distribution being independent of the distribution followed by the matrix entries might be the basis for universality which would be discussed later.

The case when the real part of entries of the matrix are all a constant c (say) and the imaginary parts are iid Normal or Uniform distribution with mean μ then the largest eigenvalue for each random matrix is distributed on a line parallel to the imaginary axis around the value $n\,c + n\,\mu\,i$ in the argand plane, where n is the size of the random matrix considered. The variation of the largest eigenvalue seems to be fixed only in the imaginary part as their real part remains fixed at $n\,c$ but their imaginary part varies slightly corresponding to the distribution of the imaginary part of each entries of the matrix and the real part remaining constant.

(a) (b)

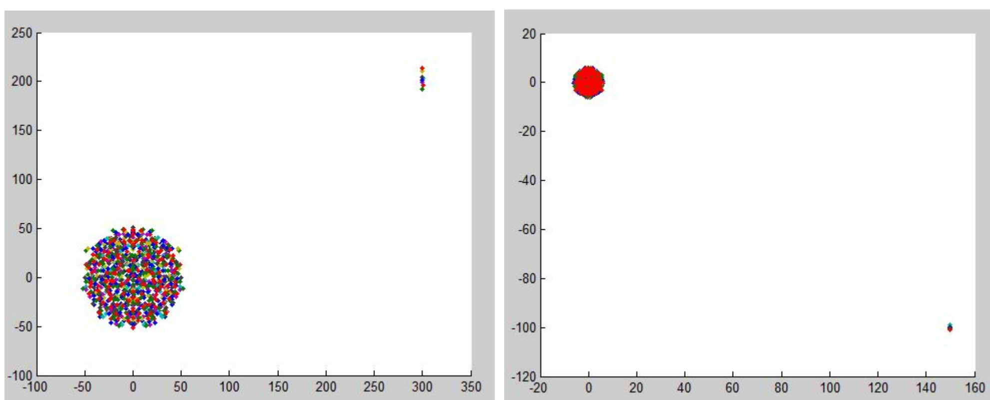

Figure 2.3: (a) Plot of eigenvalues of 10 random matrices whose entries have real part as 3 & imaginary part as iid $N(2, 5^2)$ and 100 size with central disc and the largest eigenvalues centered around (300,200). (b) Plot of eigenvalues of 10 random matrices whose entries have real part as 1.5 & imaginary part as iid $U(-2, 0)$ and 100 size with central disc and the disc of largest eigenvalues centered around (150,-100). (In both the cases the imaginary part of the largest eigenvalues seem to vary slightly while the real part remains constant)

The case when the real part of entries of the matrix are iid Normal or Uniform distribution with mean μ and the imaginary parts are all constant c (say) then the largest eigenvalue for each random matrix is distributed on a line parallel to the real axis around the value $n\,\mu + n\,c\,i$ in the argand plane, where n is the size of the random matrix considered. This variation shows that the distribution of the real part results in the variation of the real component of the eigenvalues while the imaginary component remains constant.

(a) (b)

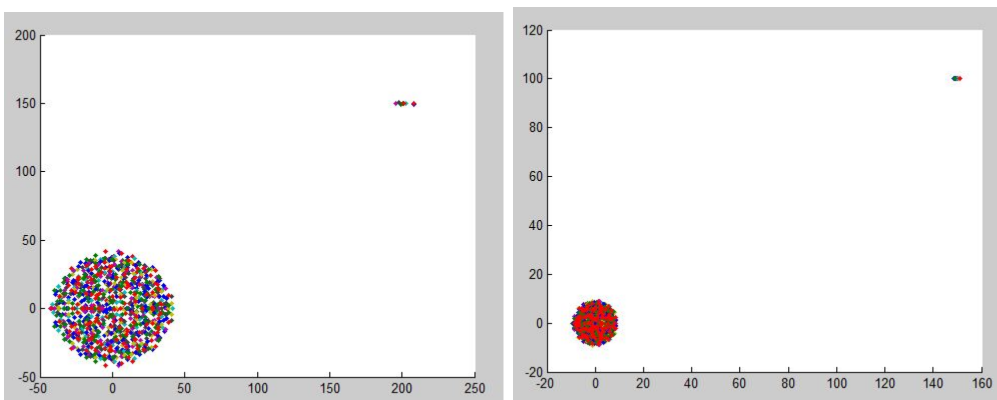

Figure 2.4: (a) Plot of eigenvalues of 10 random matrices whose entries have real part as iid $N(2,1)$ & imaginary part as 1.5 and 100 size showing the central disc and the largest eigenvalues centered around (300,200). (b) Plot of eigenvalues of 10 random matrices whose entries have real part as iid $U(0,3)$ & imaginary part as 1 and 100 size showing the central disc and the disc of largest eigenvalues centered around (150,100).

In order to illustrate the observations above MATLAB has been used to produce random matrix ensembles using random() function for producing random variables following normal or uniform distributions. The eig() function is used to compute the eigenvalues of random matrices and scatter plot has been used to plot these eigenvalues in the argand plane. A result for complex Gaussian matrix ensembles is given below proved by Ginibre [5] [1965].

Theorem 3 (Ginibre 1965,Mehta 1967)**.** *If we assume the real and imaginary parts of the entries $M_n = m_{ij}$ are i.i.d. $N(0,1)$. Then the joint distribution of the eigenvalues of M_n has density with respect to the Lebesgue measure in $\mathbb{C}^n$,*

$$p(z_1, \cdots, z_n) = \left(\pi^n \prod_{j=1}^{n} j!\right)^{-1} \exp\left(-\sum_{j=1}^{n} |z_j|^2\right) \prod_{1\le i<j\le n} |z_i - z_j|^2$$

The above theorem provides us with the joint distribution of the eigenvalues explicitly but they are not of great importance as it could not be completely related to the corresponding matrix entries and thereby an explicit relation of the distributions of the matrix entries and the eigenvalues is not achievable.
Another fact worth noticing is that the matrices are to be scaled by $\sigma\sqrt{2\,n}$ in order to restrict the spectral distribution to unit circular disc, where n is the size of the random matrix and σ is the standard deviation of the distribution of the matrix entries. Thus it becomes obvious that the radius of central spectral disc of random matrices (entries following a fixed distribution) increases with $O(n)$ (order n), where n is the size of the matrix.

Chapter 3

Complex Symmetric Matrix Ensembles With Entries Having Arbitrary Mean & Variance

In this chapter we deal with complex random matrices having entries following Normal or Uniform Distribution. As hermitian random ensembles have been widely worked upon and many results have also been established, it would be interesting to observe the behavior of complex symmetric (non-hermitian) random matrices with both real and imaginary part of entries iid Normal or Uniform distributions with arbitrary mean and variance. The results about complex random matrix and circular law have been studied by Ginibre [5], Girko [7], Mehta [10], Terence Tao [12][14], Manjunath Krishnapur [12][14] and Vu [12]. The study of circular law for complex symmetric (non-hermitian) random matrices has yielded the following observations and results. The variation of the central spectral radius for varying mean and variance has been studied. MATLAB has been used to generate random ensembles with complex entries following different probabilities and varying mean and standard deviations then finding the eigenvalues of matrices using *eig()* function. The random matrices are made complex symmetric by equating the lower diagonal entries to the upper diagonal entries.

3.1 Normally Distributed Matrix Ensembles

In order to study the distribution of the eigenvalues for random ensembles with entries following Normal distributions are considered based on the value of mean of the distribution.

3.1.1 Normally Distributed Matrix Ensembles with Zero Mean

A proof for the Circular Law has been proposed in a paper of Girko [6] but it has not yet been understood (translation from Russian might be one of the problems). For complex symmetric random matrices with real and imaginary part iid Normal distributions with mean zero and varying standard deviation the central spectral radius seems to be directly proportional to the square root of the size of the random matrix and to the standard deviation of the normal distribution. When the random matrices are normalized by the $\sigma\sqrt{2\,n}$ then the eigenvalues follow the Circular Law Conjecture, where n is the size of the random matrices & σ is the standard deviation of the normal distribution followed by each entry of the random matrix.
Let X_n be an $n \times n$ random matrix with the real and imaginary part iid Normal Distribution with mean 0 and variance σ^2 then the ESD of $\frac{1}{\sigma^2}X_n$ converges (both in probability and in the almost sure sense) to the uniform distribution on the unit disc around origin.

3.1.2 Normally Distributed Matrix Ensembles with Non-Zero Mean

For random complex symmetric matrix with entries iid normal distribution $N(\mu, \sigma^2)$ with non-zero mean μ and standard deviation σ. The observations are similar to the zero mean case with just an additional circular disc for the largest eigenvalues of each random matrix. The disc for the largest eigenvalue is centered around the point $n\mu + n\mu\, i$ in the argand plane, where n is the size of the random matrix & μ is the mean of the distribution followed by each entry. The radius of the disc consisting of the largest eigenvalues also increases linearly with increase in the standard deviation of the distribution followed by each entry of the random matrix.

3.1.3 Uniformly Distributed Matrix Ensembles

The complex symmetric random matrices with real and imaginary parts iid uniform distribution also satisfies the Circular Law. The spectral distribution for the Uniformly distributed ensembles is analogous to the spectral Normal distributed ensembles.

3.1.4 Uniform Distribution With Zero Mean

The central spectral radius is directly proportional to the square root of the size of the random matrix and to the standard deviation of the distribution.
Again, for X_n an $n \times n$ random matrix with the real and imaginary part iid Uniform Distribution with mean 0 and variance σ^2 then the ESD of $\dfrac{1}{\sigma^2}X_n$ converges (both in probability and in the almost sure sense) to the uniform distribution on the unit disc around origin.
There are not much documentation for properties about Uniformly distributed ensembles. The similarity of their spectral distribution to the Normally distributed ensembles might be one of the reasons.

3.1.5 Uniform Distribution With Non-Zero Mean

When the real and the imaginary parts are iid with Uniform distribution $U(a, b)$, where a & b (say $a < b$) with mean $\frac{a+b}{2}$ and standard deviation $\sqrt{\frac{a^2+b^2}{12}}$ then similarly the disc for the largest eigenvalue is centered around the point $n\mu + n\mu\, i$ in the argand plane, where n is the size of the random matrix & μ is the mean of the distribution followed by each entry. These various observations and results suggest an underlining *'Universality Principle'* at work as the spectral distribution seem to be dependent on the parameters of the distribution followed by each entry of the matrix rather than the nature of the distribution followed by each entry of the random matrix.

3.2 Universality And Random Matrices

This topic has been well documented in [12] even for complex symmetric ensembles the universality phenomenon is exhibited in their spectral distributions.

Theorem 4 (Central Limit Theorem)**.** *Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample of size n that is a sequence of iid random variables drawn from distribution of expected values given by μ and finite variances by σ^2. Then*

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right) \longrightarrow N(0, \sigma^2)$$

An observation analogous to the Central Limit Theorem in probability could also be observed in the cases of random ensembles where the entries are independent and identically distributed a probability distribution. Universality causes the system to be simpler, i.e. as the complexity increases then the factors on which a quantity depends becomes less complex.

(a) (b)

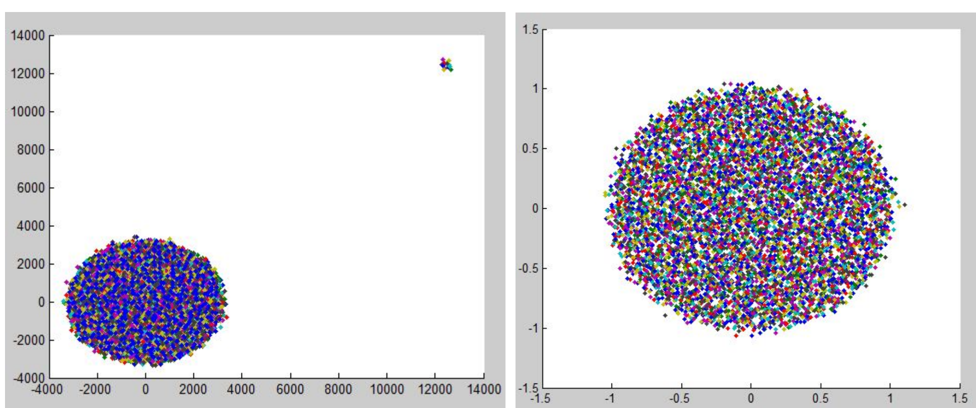

Figure 3.1: (a) Plot of eigenvalues of 15 random matrices whose entries have real part & imaginary part as iid $N(25, 100^2)$ and 500 size. (b) Plot of eigenvalues of 15 random matrices whose entries have real part & imaginary part as iid $N(25, 100^2)$ of 500 size after subtracting $M = (m_{j,k})$, $m_{j,k} = 25 + 25\,i$, $\forall\, j, k$ and then scaling the resultant by $100\sqrt{1000}$.

Proposition 2. *Let $A = (a_{j,k})_{n\times n}$ be a complex symmetric (non-hermitian) random matrix where $E[a_{j,k}] = \mu$ and $Var[a_{j,k}] = \sigma^2$, where $1 \leq j,k \leq n$ then the distribution of eigenvalues of $\dfrac{1}{\sigma\sqrt{2\,n}}\,(A - M)$, where $M = (m_{j,k})_{n\times n}$ and $m_{j,k} = (\mu + \mu\, i)$, $1 \leq j,k \leq n$ are distributed uniformly over the unit circular disc around origin.*

A similar observation could be found for entries iid uniform distribution as the proposition mentioned previously seems to be valid irrespective of the nature of distribution followed by the entries of the random matrix ensemble.

(a) (b)

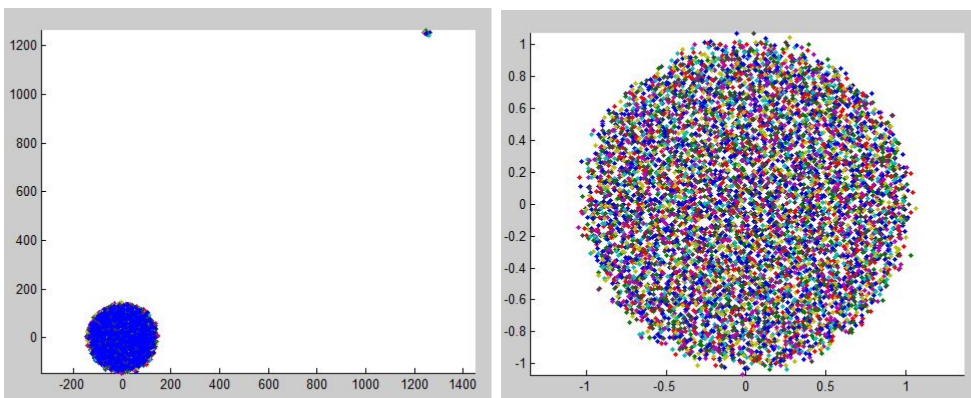

Figure 3.2: (a) Plot of eigenvalues of 15 random matrices whose entries have real part & imaginary part as iid $U(-5,10)$ and 500 size. (b) Plot of eigenvalues of 15 random matrices whose entries have real part & imaginary part as iid $U(-5,10)$ of 500 size after subtracting $M = (m_{j,k})$, $m_{j,k} = 2.5 + 2.5\,i$, $\forall\, j,k$ and then scaling the resultant by $4.3301\,\sqrt{1000}$.

Another result which indicates at some form of *Universality* at work with complex symmetric eigenvalues is the variation of the radius of the central spectral disc for random matrix ensembles with entries iid normal and uniform distributions. These observations are also true for entries iid distributions other than normal and uniform distributions. It is important to observe that the both normal & uniform distributions having the same variance seemed to be having very close (sometimes even overlapping) variation of the central spectral radius with size. The plot of

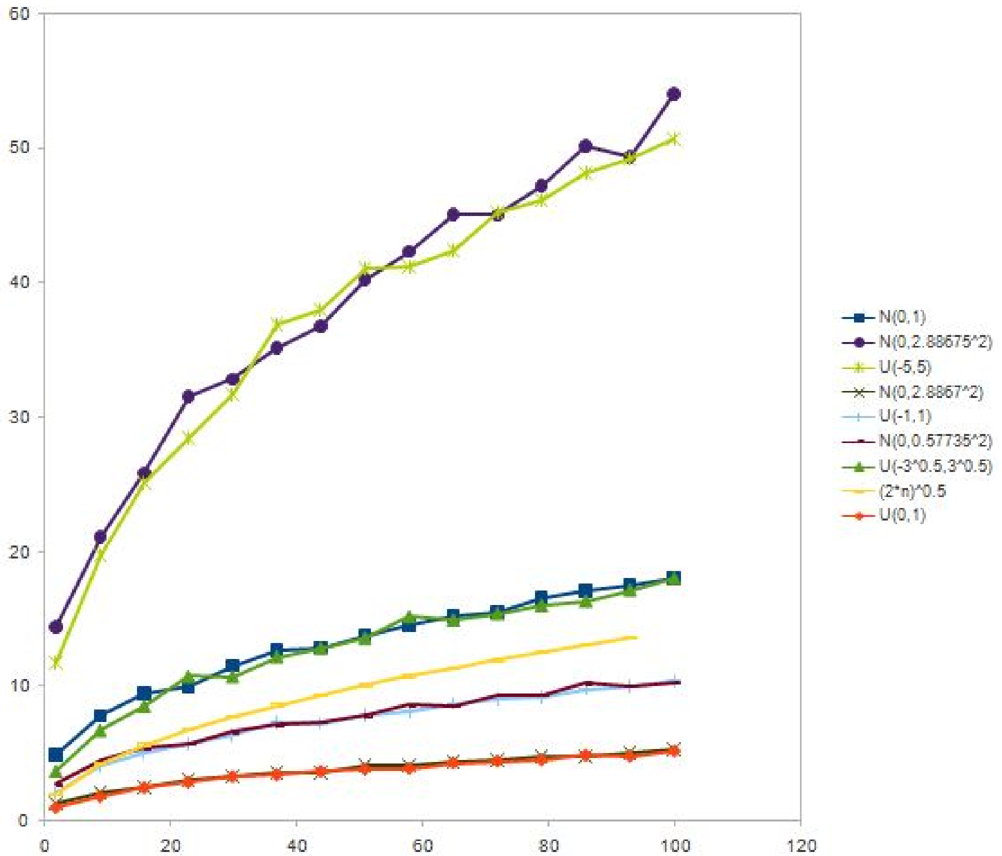

Figure 3.3: *Variation of the central spectral radius with size of the matrix*

$\sqrt{2n}$ also helps to understand the increase in the radius in every case is a multiple of it which we had mentioned earlier that if we scale the matrix by $\sigma\sqrt{2n}$ then the eigenvalues are distributed uniformly over the unit circular disc at origin.
The variation of the radius of the disc corresponding to the largest eigenvalue of each of the random matrix seems to be independent of the size of the matrix and increases with increase in standard deviation of the distribution followed by the entries for a fixed number of random matrices. This phenomenon could be explained as each random matrix has a single largest value and even though the size increases as the standard deviation and the number of random matrices are constant the radius remains approximately constant.

In order to study the properties of complex symmetric random matrices several approaches could be used to form complex symmetric random matrices. A random matrix A could be formed by using MATLAB and generating complex entries with real and imaginary part following any probability distribution then any of the following methods could be used to form a complex symmetric matrix are

- A is formed by equating $A_{i,j} = A_{j,i}$
- $B = (A + A^T)/2$
- $B = A \times A^T$

The results thus far are based on the first approach but the similar results could also be found for the other methods. The certain variations that are present are because of the variation in the parameters of the distributions due to the methods used in order to create the complex symmetric matrices. As stated earlier the spectral distributions seem to be independent of the nature of distribution of the random entries.

When the matrices are created by the second procedure then the mean of each of the entries remains the same as the mean of the entries of A but the standard deviation is altered as when $B = (A + A^T)/2$ then

$$\begin{aligned} var(B_{i,j}) &= \frac{var(A_{ij}) + var(A_{j,i})}{2^2} \qquad\qquad [\text{As } var(A_{i,j}) = var(A_{j,i})] \\ &= \frac{2\,var(A_{i,j})}{4} \\ &= \frac{var(A_{i,j})}{2} \end{aligned}$$

Similarly, when the symmetric matrices are created with the third approach as $A \times A^T$ then the observations for spectral distributions are similar but the mean and the standard deviation vary as the mean and variance are altered and could be computed for each entry using the following relations. If a, b be be independently distributed with $E[a] = \mu_a$ & $E[b] = \mu_b$a and $Var[a] = {\sigma^2}_a$ & $Var[b] = {\sigma^2}_b$ then

$$\begin{aligned} E[a+b] &= \mu_a + \mu_b \qquad & var[a+b] &= {\sigma^2}_a + {\sigma^2}_b \\ E[a\,b] &= \mu_a\,\mu_b \qquad & var[a\,b] &= {\sigma^2}_a\,{\sigma^2}_b + {\mu_a}^2\,{\sigma^2}_b + {\mu_b}^2\,{\sigma^2}_a. \quad [4] \end{aligned}$$

Chapter 4

Normally Distributed Real Matrix Ensembles

The real ensemble is also interesting and though the real entries are simpler to handle but the spectral distribution of real ensembles are complicated as the eigenvalues could be both real and imaginary. There have been studies based on which Real Gaussian Ensembles could be transformed orthogonally to hessenberg forms and from their the eigenvalue distribution could be calculated by numerical approximations.

Random matrices with real entries have been studied for the symmetric real case known as GOEs. Another approach for the study of eigenvalues could be tridiagonalization of a symmetric matrix by using Householder Rotations which are orthogonal transformations to reduce it to a tridiagonal matrix having the same eigenvalues as the symmetric matrix thereby preserving the spectral distribution.
The procedure used for tridiagonal reduction of symmetric matrices can be illustrated below.

4.1 Tridiagonalization & Hessenberg Reduction using Householder Transformations

Firstly, we consider a size n random matrix A such that

$$A = \begin{bmatrix} a_1 & b_{1,2} & b_{1,3} & \cdots & b_{1,n} \\ b_{1,2} & a_2 & b_{2,1} & \cdots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1,n} & b_{2,n} & b_{3,n} & \cdots & a_n \end{bmatrix}$$

where $a_i \sim N(0,1)$ and $b_{i,j} \sim N(0,1)$.

We can represent A as

$$A = \begin{bmatrix} a_1 & v^T \\ v & A_{n-1} \end{bmatrix} \tag{4.1.1}$$

Theorem 5. *Let $v = [x_1, x_2, \cdots, x_n]^T$ be a non-zero vector in $\mathbb{R}^n$, there exists a Householder transformation P_n of order n such that*

$$P_n\, v = \alpha\, e_1 \tag{4.1.2}$$

where $\alpha \epsilon \mathbb{R}$, $e_1 = [\,1,\, 0, \cdots, 0]$ is an elementary vector in $\mathbb{R}^n$ and $\alpha = \|v\|_2$

Therefore we can find a Householder Transformation P_1 which transforms the column vector u in (4.1.1) to a column vector $\|u\|_2\, e_1$ which is the one of the mainstays of tridiagonal and hessenberg reduction thereby aiding the process of possible numerical computations of eigenvalues of any given matrix. The procedure could be illustrated for any given symmetric matrix $A = (a_{i,j})$ then the tridiagonal reduction can be explained as

$$Q_1\,A\,{Q_1}^T = \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} a_1 & v^T \\ v & A_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \tag{4.1.3}$$

$$= \begin{bmatrix} a_1 & u^T\,P_1 \\ P_1\,u & P_1\,A_{n-1}\,P_1 \end{bmatrix} \tag{4.1.4}$$

$$= \begin{bmatrix} a_1 & c_1 & 0 & \cdots & 0 \\ c_1 & b_{11} & b_{12} & \cdots & b_{1,n-1} \\ 0 & b_{21} & b_{22} & \cdots & b_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,n-1} \end{bmatrix} \tag{4.1.5}$$

Now, the matrix $B = (b_{i,j})$ in (4.1.5) is a symmetric matrix of size $n-1$ as the matrix $P_1\,A\,P_1$ in (4.1.4) is symmetric as $A = A^T$ and we can continue the process by now considering

$$B = \begin{bmatrix} b_{11} & v_1^T \\ v_1 & B_{n-2} \end{bmatrix}$$

Again we have to find a Householder reflector P_2 which would satisfy $P_2\,v_1 = \|v_1\|_2$ and the same process could be continued as in (4.1.3) till a tridiagonal matrix is formed. We can define $P = U_{n-2}\,U_{n-3}\cdots U_1\,A\,U_1\cdots U_{n-3}\,U_{n-2}$ where

$$P\,A\,P^T = U_{n-2}\,U_{n-3}\cdots U_1\,A\,U_1\cdots U_{n-3}\,U_{n-2}$$

$$= \begin{bmatrix} d_{11} & d_{12} & 0 & 0 & \cdots & 0 \\ d_{21} & d_{22} & d_{23} & 0 & \cdots & 0 \\ 0 & d_{32} & d_{33} & d_{34} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_{n-1,n-2} & d_{n-1,n-1} & d_{n-1,n} \\ 0 & \cdots & 0 & 0 & d_{n,n-1} & d_{n,n} \end{bmatrix} \tag{4.1.6}$$

In (4.1.6) U_k is defined as

$$U_k = \begin{bmatrix} I_k & 0 \\ 0 & P_k \end{bmatrix} \ , k = 1, 2, \cdots, n-2.$$

From the above definition it follows that U_k is symmetric as P_k is symmetric. Therefore this construction implies that P is orthogonal in nature.
A similar procedure is applicable to any arbitrary matrix but in that case A in equation (4.1.1) is not reducible to the tridiagonal form as $v \neq v^T$. Thus Q can be constructed in a similar manner as earlier and hence the form in (4.1.5) becomes

$$Q\,A\,Q^T = \begin{bmatrix} a_1 & c_{12} & c_{13} & \cdots & c_{1,n} \\ c_1 & b_{11} & b_{12} & \cdots & b_{1,n-1} \\ 0 & b_{21} & b_{22} & \cdots & b_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,n-1} \end{bmatrix}$$

Therefore the tridiagonal form reduction as for the symmetric matrix case cannot be achieved. Thus an upper Hessenberg form can be computed by using similar

Householder transformations

$$P\,A\,P^T = U_{n-2}\,U_{n-3}\cdots U_1\,A\,U_1\cdots U_{n-3}\,U_{n-2}$$

$$= \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & \cdots & d_{1,n} \\ d_{21} & d_{22} & d_{23} & d_{24} & \cdots & d_{2,n} \\ 0 & d_{32} & d_{33} & d_{34} & \cdots & d_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_{n-1,n-2} & d_{n-1,n-1} & d_{n-1,n} \\ 0 & \cdots & 0 & 0 & d_{n,n-1} & d_{n,n} \end{bmatrix}$$

Though the tridiagonal and upper Hessenberg forms are not equivalent to triangular matrices which would give away the exact eigenvalues but the tridaigonal & upper Hessenberg forms use lesser storage space and easier computations of eigenvalues using numerical approximations.

4.2 Orthogonal Reduction of Normally Distributed Random Matrix Ensembles

The above mentioned concepts could prove quite useful when the spectral distributions of random matrices are considered. These could help us to compute the eigenvalues of random matrices explicitly with use of computational methods. Similar results have been stated in the notes of Manjunath Krishnapur [9] for GOE and GUE matrices in particular but in this report we generalize the facts about symmetric real matrices with entries following Normal distribution with zero mean and arbitrary variance.

Proposition 3. *Let $A = (a_{ij})_{n\times n}$ be a symmetric random matrix with all entries following standard normal distribution. If A is reduced to the tridiagonal form B by using orthogonal reduction using Householder transformations then the symmetric* *tridaigonal reduced form of A is given by*

$$B = \begin{bmatrix} c_1 & b_1 & 0 & 0 & \cdots & 0 \\ b_1 & c_2 & b_2 & 0 & \cdots & 0 \\ 0 & b_2 & c_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-1} & c_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & 0 & b_{n-1} & c_n \end{bmatrix} \tag{4.2.1}$$

where all entries of B are independently distributed and approximately

$$c_i \sim N(0,2) \textit{ for } 1 \leq i \leq n \tag{4.2.2}$$

& the sub-diagonal and super-diagonal entries

$$b_k{}^2 \sim \chi^2{}_{(n-k)} \text{ , } \textit{for } 1 \leq k \leq n-1 \tag{4.2.3}$$

The fact that the sub-diagonal and super-diagonal entries satisfy (4.2.3) as the tridiagonal reduction using Householder reflection as Householder reflection preserves the $\|.\|_2$ for any vector (column matrix). So, the square of each sub-diagonal elements are distributed following χ^2 with reducing degrees of freedom starting from $n-1$ which is the same behaviour as the sum of the squares of random numbers iid $N(0,1)$. The distribution of the of the diagonals of tridiagonal reduced form has been illustrated using distribution fitting tool *dfittool* available in MATLAB which computes the mean and the standard deviation of the distribution followed by the diagonal elements. For the normal distribution fitting in Figure 4.1 the estimated mean is 0.016697 ± 0.0642113 (error) and the standard deviation is 1.43581 ± 0.0454725 (error) which verifies the proposition given earlier as the diagonals are approximately $N(0,2)$. Note that the symmetric random matrices are constructed by equating the upper and lower triangular entries.

Now in the case of random matrices with entries following a normal distribution with mean 0 but arbitrary variance σ^2 (say) then an analogous phenomenon was

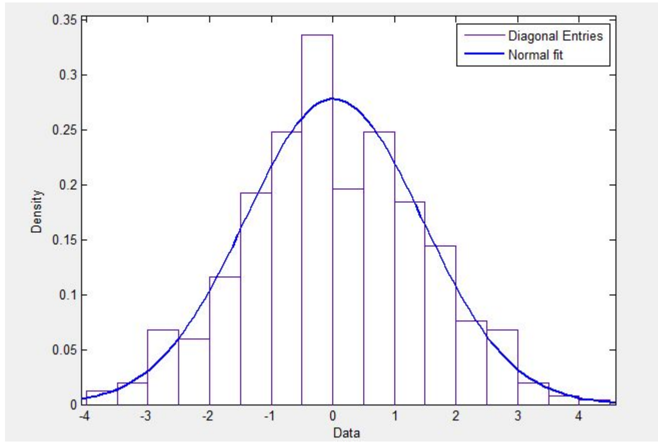

Figure 4.1: The histogram of the diagonal elements of the tridiagonal reduced form of a symmetric matrix of size 500 with all entries iid $N(0,1)$ using Householder reflections and a normal distribution fit illustrating their distribution.

observed with a corresponding change in the variance of the corresponding entries of tridiagonal form which has been put forward in the following proposition.

Proposition 4. *Let $A = (a_{ij})_{n\times n}$ be a symmetric random matrix with all entries following normal distribution with mean 0 & arbitrary variance σ^2 ($\sigma > 0$). If A is reduced to the tridiagonal form B by using orthogonal reduction using Householder transformations then the symmetric tridiagonal reduced form of A is given by*

$$B = \begin{bmatrix} c_1 & b_1 & 0 & 0 & \cdots & 0 \\ b_1 & c_2 & b_2 & 0 & \cdots & 0 \\ 0 & b_2 & c_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-1} & c_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & 0 & b_{n-1} & c_n \end{bmatrix} \tag{4.2.4}$$

where all the entries of B are independently distributed and approximately

$$c_i \sim N(0, 2\sigma^2)\ ,\ \textit{for}\ 1 \le i \le n \tag{4.2.5}$$

& the sub-diagonal and super-diagonal entries

$$b_k{}^2 \sim \textit{Non-central } \chi^2{}_{(n-k)} \text{ , } \textit{for } 1 \leq k \leq n-1 \tag{4.2.6}$$

(Non-central $\chi^2{}_{(n-k)}$ random variables could be generated by summing up squares of $n-k$ random variables iid N(0,σ^2))

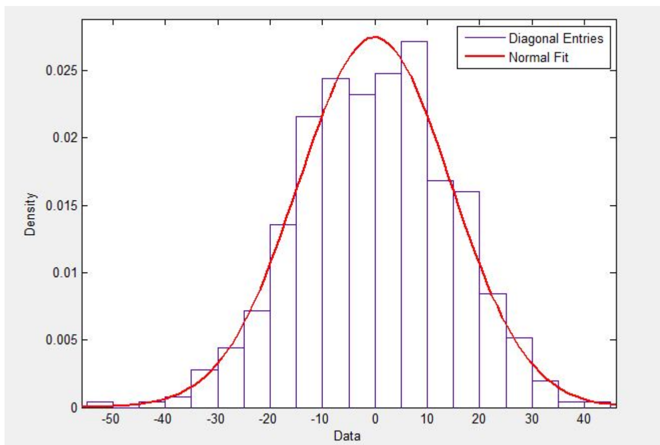

Figure 4.2: The histogram of the diagonal elements of the tridiagonal reduced form of a symmetric matrix of size 500 with all entries iid $N(0, 100)$ using Householder reflections and a normal distribution fit illustrating their distribution.

In this case also the square of each sub-diagonal elements are distributed following χ^2 with reducing degrees of freedom starting from $n-1$ which is the same behaviour as the sum of squares of random numbers iid $N(0, \sigma^2)$.

In order to generalize the results regarding orthogonal transformation of real random matrix ensembles we can consider non-symmetric real matrices with entries following $N(0, \sigma^2)$ distribution and reducing the matrix to an upper Hessenberg form using Householder transformations. The Hessenberg form could be the next best thing to getting a triangular matrix as computational methods could be applied to exact non repeated eigenvalues. For the normal distribution fitting in Figure 4.2 the estimated mean is -0.0536957 ± 0.649359 (error) and the standard deviation is

14.5201 ± 0.459856 (error) which verifies the proposition given earlier as the diagonals are approximately $N(0, 2\sigma^2)$.

Proposition 5. *Let $A = (a_{ij})_{n\times n}$ be a non-symmetric random matrix with all entries following normal distribution with mean 0 & arbitrary variance σ^2 ($\sigma > 0$). If A is reduced to the upper Hessenberg form H by using orthogonal reduction using Householder transformations then*

$$H = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & \cdots & c_{1n} \\ b_1 & c_{22} & c_{23} & c_{24} & \cdots & c_{2n} \\ 0 & b_2 & c_{33} & c_{34} & \cdots & c_{3n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-1} & c_{n-1,n-1} & c_{n-1,n} \\ 0 & \cdots & 0 & 0 & b_{n-1} & c_{n,n} \end{bmatrix} \tag{4.2.7}$$

where all the entries of H are independently distributed and approximately

$$c_{i,j} \sim N(0, \sigma^2) \text{ , for } 1 \leq i \leq n \text{ , } i \leq j \leq n \tag{4.2.8}$$

& the sub-diagonal entries

$$b_k{}^2 \sim \textit{Non-central } \chi^2{}_{(n-k)} \text{ , for } 1 \leq k \leq n-1 \tag{4.2.9}$$

The illustration in Figure 4.3 shows that the upper triangular elements $c_{i,j}$ of the upper Hessenberg form in (4.2.7) illustrates the previous proposition as the normal distribution fitting has the distribution $N(0, 60^2)$.

This proposition could enable us to calculate the eigenvalues of any given random matrix ensemble with entries following Normal Distribution with mean as 0 and any arbitrary variance. Then eigenvalues of the matrix ensemble could be obtained by reduction of the corresponding Hessenberg form to a triangular matrix thereby the diagonal entries giving away the eigenvalues.

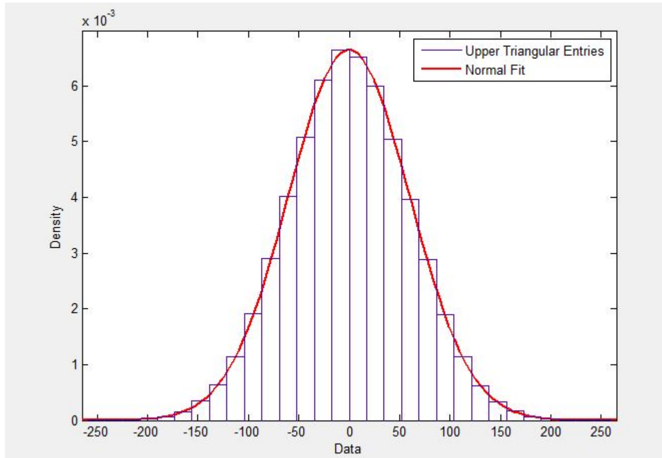

Figure 4.3: The histogram of the upper triangular elements $c_{i,j}$ of the upper Hessenberg reduced form of a matrix of size 500 with all entries iid $N(0, 60^2)$ using Householder reflections and a normal distribution fit illustrating their distribution.

But the same could not be said about random matrix ensembles with entries being Normally distributed with non-zero mean as the case when the mean of the distribution followed by the random matrix entries the distribution does not remain invariant to orthogonal transformations unlike the case when of random matrices having entries normally distributed with mean zero.
These conclusions could be drawn as the plots of the row entries illustrates the fact that the upper diagonal entries of the reduced Hessenberg form (non-zero mean) follow normal distribution but the no conclusions could be drawn regarding the of each row entries after corresponding orthogonal reduction variance and the mean. Still there is an interesting observation that the mean of the distribution of the row entries goes on decreasing to ultimately end up very close to zero as we go down the rows of the Hessenberg reduced form whereas the variance remains invariant.

Chapter 5

Future Prospects

As the field of random matrix ensembles is a vast and open field of mathematics with most of the useful work being subjected to fixed classes of random matrices (as they find a lot of applications) which could harness us towards a path of finding some results which could find useful applications. this approach is quite counter-intuitive to the conventional path of finding a class of random matrix with applications and then figuring out its properties. The generalizations of the tridiagonal & Hessenberg reduction of the real Gaussian ensembles could be used to compute and study the distribution of eigenvalues of them. One of the use of the approximation of tridiagonal & Hessenberg reduction could be in the computation of eigenvalues and the estimation of the spectral distribution of random matrices with entries iid $N(0, \sigma^2)$.
In order to illustrate an interesting observation firstly let us consider a symmetric real random matrix $A = (a_{i,j})_{n \times n}$, where $a_{i,j} \sim N(0, \sigma^2)$ and let a symmetric matrix B be of the form as in (4.1.6) where the diagonal entries are $N(0, 2\,\sigma^2)$ $(\sigma > 0)$ and the sub-diagonal and super-diagonal entries are iid Non-central ${\chi^2}_{n-k}$, where k is the the column and the row index respectively.
The above illustrations shows that though a slight change in entries of a matrix might change the eigenvalues but when we consider a random matrix A and another independently created random matrix B and study the spectral distributions of both of them then it would show that they are close enough for these matrices to have a relation between them. This fact also suggests that matrices constructed as B could be used to approximate matrices such as A thereby reducing the storage of large

(a) (b)

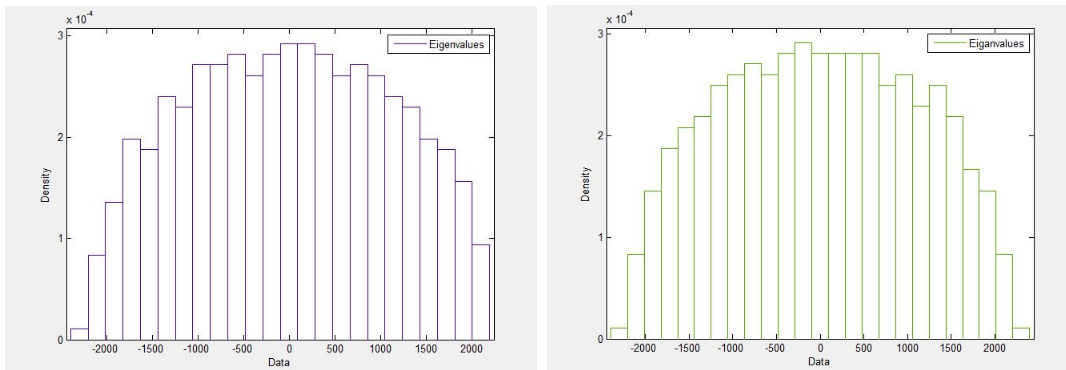

Figure 5.1: (a) Histogram of eigenvalues of random matrix of size A of size 500 and $\sigma = 50$. (b) Histogram of eigenvalues of random matrix of size B of size 500 and $\sigma = 50$.

random matrices to easily workable Tridiagonal form matrix. Similar results could be also found for a general non-symmetric matrix with entries iid $N(0, \sigma^2)$ but there we end up working with Hessenberg reduced form but it is tougher to illustrate as the eigenvalues are complex in nature. But they show similar relationship when we consider their absolute values or even real and imaginary parts.

Another interesting problem could be using the same approach of orthogonal transformations to study the properties of $A\,A^T$, where A has entries iid normal distribution. It would be tough as the entries would not follow a normal distribution and some approximation could also be necessary. But, still as many natural processes could be generalized as product of random matrices the approach could possibly lead to some fruitful work in the future.

Lastly, random matrices could also find some application in matrix completion where we could have a partially filled random matrix and try to come up with a conclusion about the exact distribution of the original matrix entries. There could be also a case where are required to eliminate noise which follows some normal distribution from a matrix system where these propositions put forward in this report could prove important. But one of the problems to the applications could be that the cases for non-zero mean are non-trivial and hard to comprehend there exact distributions.

The study of effects of arbitrary orthogonal reflectors on matrices could also draw some attention and find its application in the behaviour of random matrix ensembles where entries follow some distribution with non-zero mean.

Conclusion

Most of the work done during the span of this summer project has been based on the *Circular Law Conjecture* and the conclusions that could be drawn based on the observations and numerical experiments based on MATLAB are the following:

Proposition 1 (Circular Law Conjecture)**.** *Let X_n be the $n \times n$ random matrix whose entries are iid complex (non-hermitian) random variables with mean 0 and variance 1. Then the ESD of $\frac{1}{\sqrt{n}}X_n$ converges (both in probability and in the almost sure sense) to the uniform distribution on the unit disc around origin.*

The properties of the *Circular Law Conjecture* that have been studied with the varying properties of the matrix entries are

Proposition 2. *Let $A = (a_{j,k})_{n\times n}$ be a complex symmetric (non-hermitian) random matrix where $E[a_{j,k}] = \mu$ and $Var[a_{j,k}] = \sigma^2$, where $1 \leq j,k \leq n$ then the distribution of eigenvalues of $\dfrac{1}{\sigma\sqrt{2\,n}}\,(A - M)$, where $M = (m_{j,k})_{n\times n}$ and $m_{j,k} = (\mu + \mu\, i)$, $1 \leq j,k \leq n$ are distributed uniformly over the unit circular disc around origin.*

The conclusions for random matrices with real entries being normally distributed have been given below. The application of orthogonal reduction of to these matrices have been studied and the properties satisfied by the matrix elements are the following:

Proposition 3. *Let $A = (a_{ij})_{n\times n}$ be a symmetric random matrix with all entries following standard normal distribution. If A is reduced to the tridiagonal form B by using orthogonal reduction using Householder transformations then the symmetric tridaigonal reduced form of A is given by*

$$B = \begin{bmatrix} c_1 & b_1 & 0 & 0 & \cdots & 0 \\ b_1 & c_2 & b_2 & 0 & \cdots & 0 \\ 0 & b_2 & c_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-1} & c_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & 0 & b_{n-1} & c_n \end{bmatrix}$$

where all entries of B are independently distributed and approximately

$$c_i \sim N(0,2) \textit{ for } 1 \leq i \leq n$$

& the sub-diagonal and super-diagonal entries

$$b_k{}^2 \sim \chi^2{}_{(n-k)} \textit{ , for } 1 \leq k \leq n-1$$

Proposition 4. *Let $A = (a_{ij})_{n \times n}$ be a non-symmetric random matrix with all entries following normal distribution with mean 0 & arbitrary variance σ^2 ($\sigma > 0$). If A is reduced to the upper Hessenberg form H by using orthogonal reduction using Householder transformations then*

$$H = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & \cdots & c_{1n} \\ b_1 & c_{22} & c_{23} & c_{24} & \cdots & c_{2n} \\ 0 & b_2 & c_{33} & c_{34} & \cdots & c_{3n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-1} & c_{n-1,n-1} & c_{n-1,n} \\ 0 & \cdots & 0 & 0 & b_{n-1} & c_{n,n} \end{bmatrix}$$

where all the entries of H are independently distributed and approximately

$$c_{i,j} \sim N(0,\sigma^2) \textit{ , for } 1 \leq i \leq n \textit{ , } i \leq j \leq n$$

& the sub-diagonal entries

$$b_k{}^2 \sim \textit{Non-central } \chi^2{}_{(n-k)} \textit{ , for } 1 \leq k \leq n-1$$

The propositions could be used to study and approximate eigenvalues of random matrix ensembles. One of the useful results could also be the eigenvalue distribution exhibited by the tridiagonal random matrices with each non-zero entry following χ^2 distribution. The distribution seems to be very similar to the eigenvalue distribution of a random matrix with real Gaussian entries.

Schedule

Week 1

Study of Semi-circle Law and Circular Law with MATLAB Codes.

Week 2

Study of the relation between mean and variance of random matrix elements and the spectral radius of the random matrices.

Week 3

Study of complex symmetric (non-hermitian) random matrices with entries following normal or uniform distribution.

Week 4

Establishing a relation between the spectral radius and the size of the random matrix, mean & variance of the matrix elements. Illustrations of the universality phenomenon in the case of distribution of spectral radius of random matrices.

Week 5

Study of Householder's Transformations for orthogonal reduction of random matrices with the entries being real and following normal distribution.

Week 6

Tridiagonalization reduction of symmetric random matrices with entries following normal distribution.

Week 7

Hessenberg reduction of non-symmetric random matrices with entries following normal distribution.

Week 8

Studying the behaviour of Hessenberg reduction of random matrices with real entries following normal distribution with non-zero mean.

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